

# On small matrix subalgebras with a trivial centralizer \*

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## Abstract

Given an integer  $n \geq 3$ , we investigate the minimal dimension of a subalgebra of  $M_n(\mathbb{K})$  with a trivial centralizer. It is shown that this dimension is 5 when  $n$  is even and 4 when it is odd. In the latter case, we also determine all 4-dimensional subalgebras with a trivial centralizer.

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## 1 Introduction

Here, we set an integer  $n \geq 2$  and a field  $\mathbb{K}$ . Using the French convention, we let  $\mathbb{N}$  denote the set of non-negative integers and  $\mathbb{N}^*$  the one of positive integers. We let  $M_n(\mathbb{K})$  denote the algebra of square matrices of order  $n$  with entries in  $\mathbb{K}$  and  $T_n(\mathbb{K})$  its subalgebra of upper triangular matrices. We let  $M_{p,q}(\mathbb{K})$  denote the set of matrices with  $p$  rows,  $q$  columns and entries in  $\mathbb{K}$ .

All subalgebras of  $M_n(\mathbb{K})$  are required to contain the unit matrix  $I_n$ : the subalgebra  $\text{Span}(I_n)$  will be called **trivial**.

For  $(i, j) \in \llbracket 1, n \rrbracket^2$ , we let  $E_{i,j}$  denote the elementary matrix of  $M_n(\mathbb{K})$  with a zero entry in every position except for  $(i, j)$  where the entry is 1.

Given a subset  $V$  of  $M_n(\mathbb{K})$ , we let

$$\mathcal{C}(V) := \{A \in M_n(\mathbb{K}) : \forall M \in V, AM = MA\}$$

denote its **centralizer**, and we simply write  $\mathcal{C}(A) = \mathcal{C}(\{A\})$  when  $A$  is a matrix of  $M_n(\mathbb{K})$ . Recall that  $\mathcal{C}(V)$  is always a subalgebra of  $M_n(\mathbb{K})$  and that  $\mathcal{C}(V)$  is

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also the centralizer of the subalgebra generated by  $V$ .

The Jordan matrix of order  $n$  will be written  $J_n = (\delta_{i+1,j})_{1 \leq i,j \leq n}$ , where  $\delta_{a,b}$  equals 1 if  $a = b$ , and 0 otherwise.

In this paper, we will focus on subalgebras of  $M_n(\mathbb{K})$  with a small dimension and a trivial centralizer. The basic motivation for studying subalgebras with a trivial centralizer comes from the theory of representations of algebras. Let  $\mathcal{A}$  be a subalgebra of  $M_n(\mathbb{K})$ , which we identify with the algebra of linear endomorphisms of the vector space  $\mathbb{K}^n$ . This yields a structure of  $\mathcal{A}$ -module on  $\mathbb{K}^n$  for which the endomorphisms naturally correspond to the matrices in the centralizer  $\mathcal{C}(\mathcal{A})$ . When  $\mathbb{K}$  is algebraically closed and  $\mathbb{K}^n$  is a simple  $\mathcal{A}$ -module (i.e. when it has no non-trivial submodule), then  $\mathcal{C}(\mathcal{A})$  is trivial, however the converse may not hold. In the case  $\mathcal{A}$  is generated by a finite subgroup of  $GL_n(\mathbb{K})$  and  $\mathbb{K}$  has characteristic 0, then the converse is classically true because  $\mathcal{A}$  is then semi-simple (see the theory of linear representations of finite groups). In the general case of an arbitrary field and an arbitrary subalgebra of  $M_n(\mathbb{K})$ , the condition  $\mathcal{C}(\mathcal{A}) = \text{Span}(I_n)$  may thus be seen as an alternative notion of simplicity or semi-simplicity, which provides motivation enough for studying it systematically.

Non-trivial subalgebras of  $M_n(\mathbb{K})$  with a trivial centralizer are actually commonplace. A classical example is that of  $T_n(\mathbb{K})$ . Indeed, let  $A \in \mathcal{C}(T_n(\mathbb{K}))$ . Then  $A$  commutes with  $E_{i,i}$  for every  $i \in \llbracket 1, n \rrbracket$ , which shows that  $A$  is diagonal. The commutation of  $A$  with  $E_{1,i}$  for every  $i \in \llbracket 2, n \rrbracket$  then shows that all diagonal entries of  $A$  are equal.

It is somewhat harder to produce such subalgebras with a small dimension. Our main goal is to find the smallest dimension for such a subalgebra and to classify the subalgebras of minimal dimension.

**Definition 1.** We let  $t_n(\mathbb{K})$  (or simply  $t_n$  when the field is obvious) denote the smallest dimension of a subalgebra of  $M_n(\mathbb{K})$  with a trivial centralizer.

Notice first that a subalgebra of dimension 2 is always of the form  $\mathbb{K}[A]$  for some matrix  $A$  which is not a scalar multiple of  $I_n$ , so its centralizer contains  $A$  and is therefore non-trivial. This essentially solves the case  $n = 2$ .

**Proposition 1.** *One has  $t_2 = 3$ . More precisely,  $T_2(\mathbb{K})$  has a trivial centralizer and dimension 3.*

We will assume  $n \geq 3$  from now on. Our main results are stated below:

**Theorem 2.** *If  $n \geq 3$  is even, then  $t_n(\mathbb{K}) = 5$ .  
If  $n \geq 3$  is odd, then  $t_n(\mathbb{K}) = 4$ .*

**Proposition 3.** *Let  $p \in \mathbb{N} \setminus \{0, 1\}$  and consider the linear subspace  $\mathcal{F}_{2p}$  of  $M_{2p}(\mathbb{K})$  generated by the matrices*

$$\begin{bmatrix} I_p & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & I_p \end{bmatrix}, \begin{bmatrix} 0 & I_p \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & J_p \\ 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & J_p^t \\ 0 & 0 \end{bmatrix}.$$

*Then  $\mathcal{F}_{2p}$  is a 5-dimensional subalgebra of  $M_{2p}(\mathbb{K})$  with a trivial centralizer.*

**Proposition 4.** *Let  $p \in \mathbb{N}^*$ . Consider the matrices  $C_p = \begin{bmatrix} I_p & 0 \\ 0 & I_p \end{bmatrix}$  and  $D_p = \begin{bmatrix} I_p & 0 \\ 0 & I_p \end{bmatrix}$  in  $M_{p,p+1}(\mathbb{K})$ , and define  $\mathcal{H}_{2p+1}$  as the linear subspace of  $M_{2p+1}(\mathbb{K})$  generated by the matrices*

$$\begin{bmatrix} I_p & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & I_{p+1} \end{bmatrix}, \begin{bmatrix} 0 & C_p \\ 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & D_p \\ 0 & 0 \end{bmatrix}.$$

*Then  $\mathcal{H}_{2p+1}$  is a 4-dimensional subalgebra of  $M_{2p+1}(\mathbb{K})$  with a trivial centralizer.*

Finally, we will prove that the latter example is essentially unique:

**Proposition 5.** *Let  $p \in \mathbb{N}^*$  and  $\mathcal{A}$  be a subalgebra of  $M_{2p+1}(\mathbb{K})$  of dimension 4 with a trivial centralizer. Then  $\mathcal{A}$  is **conjugate** to either  $\mathcal{H}_{2p+1}$  or its transposed subalgebra  $\mathcal{H}_{2p+1}^t$ , i.e. there is a non-singular matrix  $P \in GL_{2p+1}(\mathbb{K})$  such that*

$$\mathcal{A} = P \mathcal{H}_{2p+1} P^{-1} \quad \text{or} \quad \mathcal{A} = P \mathcal{H}_{2p+1}^t P^{-1}.$$

*Remark 1.* It is an easy exercise to prove that  $\mathcal{H}_{2p+1}$  and  $\mathcal{H}_{2p+1}^t$  are not conjugate one to the other.

## 2 Checking the examples

We will start with a little lemma.

**Lemma 6.** *Let  $n \geq 2$  be an integer. Then  $\text{Span}(J_n, J_n^t)$  has a trivial centralizer.*

*Proof.* Since  $J_n$  is cyclic, its centralizer is  $\mathbb{K}[J_n]$  (see [5] Theorem 5 p.23), and it thus contains only upper triangular matrices with equal diagonal entries. Similarly, every matrix of  $\mathbb{K}[J_n^t]$  is lower triangular. It follows that every matrix in the centralizer of  $\text{Span}(J_n, J_n^t)$  must be scalar.  $\square$

The examples featured in Propositions 3 and 4 are based upon the same idea. Consider a decomposition  $n = p + q$  and a linear subspace  $V$  of  $M_{p,q}(\mathbb{K})$ . It is then easily checked that

$$\mathcal{H} := \left\{ \begin{bmatrix} a \cdot I_p & K \\ 0 & b \cdot I_q \end{bmatrix} \mid (a, b) \in \mathbb{K}^2, K \in V \right\}$$

is always a subalgebra of  $M_n(\mathbb{K})$  with dimension  $\dim V + 2$ . Straightforward computation also shows that the centralizer of the matrix  $P = \begin{bmatrix} I_p & 0 \\ 0 & 0 \end{bmatrix}$  is

$$\mathcal{C}(P) = \left\{ \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \mid (X, Y) \in M_p(\mathbb{K}) \times M_q(\mathbb{K}) \right\}.$$

Let  $M = \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \in \mathcal{C}(P)$ . Then  $M$  belongs to  $\mathcal{C}(\mathcal{H})$  if and only if  $\forall K \in V, XK = KY$ . Of course, this last relation need only be tested on a basis of  $V$ .

From there, our claims may easily be proven.

**The example in Proposition 3.**

Here,  $q = p$  and  $V = \text{Span}(I_p, J_p, J_p^t)$ .

Let  $(X, Y) \in M_p(\mathbb{K})^2$  such that  $XI_p = I_pY$ ,  $XJ_p = J_pY$  and  $XJ_p^t = J_p^tY$ . Then  $Y = X$  and  $X$  commutes with both  $J_p$  and  $J_p^t$ . By Lemma 6,  $X$  is a scalar multiple of  $I_p$  hence  $\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}$  is a scalar multiple of  $I_n$ . This proves Proposition 3.

**The example in Proposition 4.**

Here  $q = p + 1$  and  $V = \text{Span}(C_p, D_p)$ .

Let  $(X, Y) \in M_p(\mathbb{K}) \times M_{p+1}(\mathbb{K})$  such that  $\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \in \mathcal{C}(\mathcal{H})$ .

The identity  $XC_p = C_pY$  entails that  $Y = \begin{bmatrix} X & 0 \\ L & \alpha \end{bmatrix}$  for some  $(\alpha, L) \in \mathbb{K} \times M_{1,p}(\mathbb{K})$ , whilst identity  $XD_p = D_pY$  shows that  $Y = \begin{bmatrix} \beta & L' \\ 0 & X \end{bmatrix}$  for some  $(\beta, L') \in \mathbb{K} \times M_{1,p}(\mathbb{K})$ . We thus have

$$\begin{bmatrix} X & 0 \\ L & \alpha \end{bmatrix} = \begin{bmatrix} \beta & L' \\ 0 & X \end{bmatrix}.$$

Starting from the first column of  $X$ , an easy induction shows that  $X$  is upper triangular with all diagonal entries equal to  $\beta$ . Also, starting from the last column of  $X$ , an easy induction shows that  $L' = 0$  and  $X$  is lower triangular. This yields  $X = \beta.I_p$  and  $Y = \beta.I_{p+1}$  hence  $\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} = \beta.I_{2p+1}$ , which proves Proposition 4.

### 3 A lower bound for $t_n(\mathbb{K})$

#### 3.1 Introduction

Here, we will prove that  $t_n \geq 4$  when  $n$  is odd, and  $t_n \geq 5$  when  $n$  is even. In other words, we will prove that every subalgebra of  $M_n(\mathbb{K})$  has a non-trivial centralizer provided it has dimension  $p \leq 3$  when  $n$  is odd, and dimension  $p \leq 4$  when  $p$  is even. The proof is essentially laid out as follows:

- by extending the ground field, we reduce the study to the case of an algebraically closed field;
- in this case, we discard all *unispectral* subalgebras (i.e. subalgebras in which every operator has a sole eigenvalue);
- in the remaining cases, the considered subalgebra contains a non-trivial idempotent which we use to split the algebra  $\mathcal{A}$  into several remarkable subspaces; we then use that splitting to find a non-scalar matrix in the centralizer of  $\mathcal{A}$  when  $\dim \mathcal{A}$  is small enough.

From now on, we set an integer  $n \geq 3$  and a subalgebra  $\mathcal{A}$  of  $M_n(\mathbb{K})$ . The following elementary facts will be used repeatedly:

- for every  $P \in GL_n(\mathbb{K})$ , the conjugate subalgebra  $P\mathcal{A}P^{-1}$  has the same dimension as  $\mathcal{A}$  and its centralizer  $P\mathcal{C}(\mathcal{A})P^{-1}$  is trivial if and only if  $\mathcal{C}(\mathcal{A})$  is trivial.
- the transposed subalgebra  $\mathcal{A}^t := \{M^t \mid M \in \mathcal{A}\}$  has the same dimension as  $\mathcal{A}$  and its centralizer  $\mathcal{C}(\mathcal{A})^t$  is trivial if and only if  $\mathcal{C}(\mathcal{A})$  is trivial.

#### 3.2 Reduction to the case of an algebraically closed field

Let  $\mathbb{L}$  be a field extension of  $\mathbb{K}$ . Recall that when  $\mathcal{A}$  is a subalgebra of  $M_n(\mathbb{K})$  and we let  $\mathcal{A}_{\mathbb{L}}$  denote the linear  $\mathbb{L}$ -subspace of  $M_n(\mathbb{L})$  generated by  $\mathcal{A}$ , then

$\mathcal{A}_{\mathbb{L}}$  is a subalgebra of  $M_n(\mathbb{L})$ . The natural isomorphism of  $\mathbb{L}$ -algebras  $M_n(\mathbb{L}) \simeq M_n(\mathbb{K}) \otimes_{\mathbb{K}} \mathbb{L}$  maps  $\mathcal{A}_{\mathbb{L}}$  to  $\mathcal{A} \otimes_{\mathbb{K}} \mathbb{L}$  hence  $\dim_{\mathbb{L}} \mathcal{A}_{\mathbb{L}} = \dim_{\mathbb{K}} \mathcal{A}$ . Also, the centralizer of  $\mathcal{A} \otimes_{\mathbb{K}} \mathbb{L}$  in  $M_n(\mathbb{K}) \otimes_{\mathbb{K}} \mathbb{L}$  is clearly  $\mathcal{C}(\mathcal{A}) \otimes_{\mathbb{K}} \mathbb{L}$ , therefore  $\mathcal{A}$  has a trivial centralizer in  $M_n(\mathbb{K})$  if and only if  $\mathcal{A}_{\mathbb{L}}$  has a trivial centralizer in  $M_n(\mathbb{L})$ . We deduce that

$$t_n(\mathbb{K}) \geq t_n(\mathbb{L}).$$

Therefore, by Steinitz's theorem and the examples discussed earlier, it will suffice to prove theorem 2 when  $\mathbb{K}$  is algebraically closed.

In the rest of this section, we assume  $\mathbb{K}$  is algebraically closed.

### 3.3 The case $\mathcal{A}$ is unispectral

**Definition 2.** We call a matrix  $A \in M_n(\mathbb{K})$  **unispectral** when it has a sole eigenvalue.

A subalgebra of  $M_n(\mathbb{K})$  is called unispectral when all its elements are unispectral.

The standard example is the subalgebra of matrices of the form  $\lambda I_n + T$  for some strictly upper triangular matrix  $T$ . Conversely, every unispectral subalgebra is conjugate to a subalgebra of the preceding one:

**Proposition 7.** *Let  $\mathcal{A}$  be a unispectral subalgebra of  $M_n(\mathbb{K})$ . Then there is a non-singular  $P \in GL_n(\mathbb{K})$  such that  $PAP^{-1} \subset T_n(\mathbb{K})$ .*

*Proof.* We use Burnside's theorem (see [3] p.213) by induction on  $n$ . The case  $n = 1$  is trivial. Assume  $n \geq 2$  and our claim holds for every non-negative integer  $p < n$  and every unispectral subalgebra of  $M_p(\mathbb{K})$ . Let  $\mathcal{A}$  be a unispectral subalgebra of  $M_n(\mathbb{K})$ . Clearly,  $\mathcal{A} \subsetneq M_n(\mathbb{K})$  hence Burnside's theorem shows there is a non-singular  $P \in GL_n(\mathbb{K})$  and an integer  $p \in \llbracket 1, n-1 \rrbracket$  such that every  $M \in \mathcal{A}$  splits as

$$M = P^{-1} \begin{bmatrix} A(M) & * \\ 0 & B(M) \end{bmatrix} P \quad \text{where } A(M) \in M_p(\mathbb{K}) \text{ and } B(M) \in M_{n-p}(\mathbb{K}).$$

Then  $\mathcal{A}_1 := \{A(M) \mid M \in \mathcal{A}\}$  (resp.  $\mathcal{A}_2 := \{B(M) \mid M \in \mathcal{A}\}$ ) is a unispectral subalgebra of  $M_p(\mathbb{K})$  (resp. of  $M_{n-p}(\mathbb{K})$ ). Using the induction hypothesis, there are non-singular matrices  $P_1 \in GL_p(\mathbb{K})$  and  $P_2 \in GL_{n-p}(\mathbb{K})$  such that  $P_1 \mathcal{A}_1 P_1^{-1} \subset T_p(\mathbb{K})$  and  $P_2 \mathcal{A}_2 P_2^{-1} \subset T_{n-p}(\mathbb{K})$ . Setting  $Q := \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}$ , we have  $Q \in GL_n(\mathbb{K})$  and  $(QP)\mathcal{A}(QP)^{-1} \subset T_n(\mathbb{K})$ .  $\square$

**Corollary 8.** *Let  $p \geq 2$  be an integer. Then every unispectral subalgebra of  $M_p(\mathbb{K})$  has a non-trivial centralizer.*

*Proof.* It suffices to prove the statement for any unispectral subalgebra  $\mathcal{A}$  of  $T_p(\mathbb{K})$ . However any matrix  $M$  of  $\mathcal{A}$  must have identical diagonal entries and must be upper triangular: it easily follows that the elementary matrix  $E_{1,n}$  lies in  $\mathcal{C}(\mathcal{A})$ .  $\square$

In what follows, we will assume  $\mathcal{A}$  is not unispectral.

### 3.4 The basic splitting

Let us choose a non-unispectral matrix  $M \in \mathcal{A}$ . Choose then a spectral projection  $P$  associated to  $M$ : hence  $P \in \mathbb{K}[M] \subset \mathcal{A}$  (see [1] chapter 8, 4 corollary 3 p.271) and we have therefore found a non-trivial idempotent in  $\mathcal{A}$ . By conjugating  $\mathcal{A}$  with an appropriate non-singular matrix, we are reduced to the case  $\mathcal{A}$  contains, for some  $p \in \llbracket 1, n-1 \rrbracket$ , the idempotent matrix

$$P := \begin{bmatrix} I_p & 0 \\ 0 & 0 \end{bmatrix} \in M_n(\mathbb{K}).$$

*Remark 2.* The rest of the proof will only rely on this assumption and the condition on the dimension of  $\mathcal{A}$ . In particular, the reader will check that it does not use the fact that  $\mathbb{K}$  is algebraically closed.

In what follows, we set  $q := n - p$ . Notice then that  $\mathcal{A}$  also contains  $Q := I_n - P = \begin{bmatrix} 0 & 0 \\ 0 & I_q \end{bmatrix}$ . Since  $\mathcal{A}$  is a subalgebra containing both  $P$  and  $Q$ , one clearly has:

$$\begin{aligned} P\mathcal{A}P &= \left\{ \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \mid M \in M_p(\mathbb{K}) \right\} \cap \mathcal{A} \quad , \quad P\mathcal{A}Q = \left\{ \begin{bmatrix} 0 & M \\ 0 & 0 \end{bmatrix} \mid M \in M_{p,q}(\mathbb{K}) \right\} \cap \mathcal{A}, \\ Q\mathcal{A}P &= \left\{ \begin{bmatrix} 0 & 0 \\ M & 0 \end{bmatrix} \mid M \in M_{q,p}(\mathbb{K}) \right\} \cap \mathcal{A}, \quad \text{and} \quad Q\mathcal{A}Q = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & M \end{bmatrix} \mid M \in M_q(\mathbb{K}) \right\} \cap \mathcal{A}. \end{aligned}$$

Those four linear subspaces of  $\mathcal{A}$  will respectively be denoted by  $\mathcal{A}_{1,1}$ ,  $\mathcal{A}_{1,2}$ ,  $\mathcal{A}_{2,1}$  and  $\mathcal{A}_{2,2}$ . For every  $M \in M_n(\mathbb{K})$ , write:

$$M = I_n M I_n = (P + Q) M (P + Q) = P M P + P M Q + Q M P + Q M Q.$$

It follows that

$$\mathcal{A} = \mathcal{A}_{1,1} \oplus \mathcal{A}_{1,2} \oplus \mathcal{A}_{2,1} \oplus \mathcal{A}_{2,2}$$

hence

$$\dim \mathcal{A} = \dim \mathcal{A}_{1,1} + \dim \mathcal{A}_{1,2} + \dim \mathcal{A}_{2,1} + \dim \mathcal{A}_{2,2}.$$

Notice also that  $P \in \mathcal{A}_{1,1}$  and  $Q \in \mathcal{A}_{2,2}$ , so that

$$\dim \mathcal{A}_{1,1} \geq 1 \quad \text{and} \quad \dim \mathcal{A}_{2,2} \geq 1.$$

In what follows, we will discuss various cases depending on the respective dimensions of the  $\mathcal{A}_{i,j}$ 's. One case can readily be done away: if  $\mathcal{A}_{1,2} = \{0\}$  and  $\mathcal{A}_{2,1} = \{0\}$ , then  $P$  belongs to  $\mathcal{C}(\mathcal{A}) \setminus \text{Span}(I_n)$ .

We will now assume  $\mathcal{A}_{1,2} \neq \{0\}$  or  $\mathcal{A}_{2,1} \neq \{0\}$ .

### 3.5 The case $\dim \mathcal{A} = 3$

It only remains to investigate the case one of  $\mathcal{A}_{2,1}$  and  $\mathcal{A}_{1,2}$  has dimension 1 and the other 0. Transposition of  $\mathcal{A}$  only leaves us with the case  $\mathcal{A}_{2,1} = \{0\}$  and  $\mathcal{A}_{1,2}$  is generated by some  $C \in \text{M}_{p,q}(\mathbb{K}) \setminus \{0\}$ . Hence  $\mathcal{A}_{1,1} = \text{Span}(I_p)$  and  $\mathcal{A}_{2,2} = \text{Span}(I_q)$ , so the considerations of Section 2 show that, in order to prove that  $\mathcal{C}(\mathcal{A})$  is non-trivial, it will suffice to prove that, for some pair  $(X, Y) \in \text{M}_p(\mathbb{K}) \times \text{M}_q(\mathbb{K})$ , one has  $XC = CY$  and  $\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}$  is not a scalar multiple of  $I_n$ . Consider then the linear map:

$$f : \begin{cases} \text{M}_p(\mathbb{K}) \times \text{M}_q(\mathbb{K}) & \longrightarrow \text{M}_{p,q}(\mathbb{K}) \\ (X, Y) & \longmapsto XC - CY. \end{cases}$$

By the rank theorem:

$$\dim \text{Ker } f \geq \dim(\text{M}_p(\mathbb{K}) \times \text{M}_q(\mathbb{K})) - \dim \text{M}_{p,q}(\mathbb{K}) = p^2 + q^2 - pq = (p-q)^2 + pq \geq pq.$$

Since  $n \geq 3$  and  $p \in \llbracket 1, n-1 \rrbracket$ , one has  $pq \geq 2$ , which shows that  $\text{Ker } f \neq \text{Span}((I_p, I_q))$ . Therefore,  $\mathcal{C}(\mathcal{A})$  is non-trivial.



### 3.6 The case $\dim \mathcal{A} = 4$

We now assume  $\dim \mathcal{A} = 4$  and set

$$\nu(\mathcal{A}) = (\dim \mathcal{A}_{1,1}, \dim \mathcal{A}_{1,2}, \dim \mathcal{A}_{2,1}, \dim \mathcal{A}_{2,2}).$$

Transposition and conjugation by a permutation matrix help us reduce the situation to only three cases:

- $\nu(\mathcal{A}) = (2, 1, 0, 1)$ ;
- $\nu(\mathcal{A}) = (1, 1, 1, 1)$ ;
- $\nu(\mathcal{A}) = (1, 2, 0, 1)$ .

In the first two cases, we will show that  $\mathcal{C}(\mathcal{A})$  is non-trivial, even when  $n$  is odd. In the last case, we will show that  $\mathcal{C}(\mathcal{A})$  is non-trivial when  $n$  is even.

#### 3.6.1 The case $\nu(\mathcal{A}) = (2, 1, 0, 1)$ .

In this case, there is some  $C \in M_p(\mathbb{K}) \setminus \text{Span}(I_p)$  and some  $V \in M_{p,q}(\mathbb{K}) \setminus \{0\}$  such that  $\mathcal{A}$  is generated by the matrices

$$\begin{bmatrix} I_p & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & I_q \end{bmatrix}, \begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & V \\ 0 & 0 \end{bmatrix}.$$

Since the product of the last two matrices belongs to  $\mathcal{A}$ , one must have  $C V = \lambda V$  for some  $\lambda \in \mathbb{K}$ . Replacing  $C$  with  $C - \lambda I_p$ , we may assume  $C V = 0$ , in which case the matrix  $\begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix}$  clearly belongs to  $\mathcal{C}(\mathcal{A}) \setminus \text{Span}(I_n)$ .

#### 3.6.2 The case $\nu(\mathcal{A}) = (1, 1, 1, 1)$ .

In this case, there are matrices  $U \in M_{q,p}(\mathbb{K}) \setminus \{0\}$  and  $V \in M_{p,q}(\mathbb{K}) \setminus \{0\}$  such that  $\mathcal{A}$  is generated by the four matrices

$$\begin{bmatrix} I_p & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & I_q \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ U & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & V \\ 0 & 0 \end{bmatrix}.$$

A matrix belongs to  $\mathcal{C}(\mathcal{A})$  if and only if it has the form  $\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}$ , where  $(X, Y) \in M_p(\mathbb{K}) \times M_q(\mathbb{K})$  satisfies  $UX = YU$  and  $XV = VY$ .

One obvious element in this centralizer is  $\begin{bmatrix} VU & 0 \\ 0 & UV \end{bmatrix}$ . Assume now that  $\mathcal{C}(\mathcal{A})$  is trivial. There would then exist some  $\lambda \in \mathbb{K}$  such that  $VU = \lambda I_p$  and  $UV = \lambda I_q$ . Two situations may arise:

- The case  $\lambda \neq 0$ . Replacing  $V$  by  $\frac{1}{\lambda}V$ , we may assume  $\lambda = 1$ . Then standard rank considerations show that  $p = q$  and  $V = U^{-1}$ . Equality  $XV = VY$  thus implies  $UX = YU$ , hence  $\mathcal{A}$  has the same centralizer as the subalgebra generated by  $P$ ,  $Q$  and  $\begin{bmatrix} 0 & V \\ 0 & 0 \end{bmatrix}$ , which has been shown to be non-trivial in Section 3.5. There lies a contradiction.
- The case  $\lambda = 0$ . Then  $UV = 0$  and  $VU = 0$ . Since neither  $U$  nor  $V$  is zero, we deduce that  $\text{Ker } U$  and  $\text{Im } V$  are non-trivial subspaces of  $\mathbb{K}^p$  and  $\text{Ker } V$  and  $\text{Im } U$  are non-trivial subspaces of  $\mathbb{K}^q$ . We can therefore construct rank 1 matrices  $X \in M_p(\mathbb{K})$  and  $Y \in M_q(\mathbb{K})$  such that  $\text{Im } X \subset \text{Ker } U$ ,  $\text{Im } V \subset \text{Ker } X$ ,  $\text{Im } Y \subset \text{Ker } V$  and  $\text{Im } U \subset \text{Ker } Y$ . Hence  $XV = VY = 0$  and  $UX = YU = 0$ , and it follows that the block diagonal matrix  $\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}$  belongs to  $\mathcal{C}(\mathcal{A}) \setminus \text{Span}(I_n)$ , another contradiction.

The previous *reductio ad absurdum* then proves that  $\mathcal{C}(\mathcal{A})$  is non-trivial.

### 3.6.3 The case $\nu(\mathcal{A}) = (1, 2, 0, 1)$ .

We actually lose no generality assuming  $p \leq q$  (if not, we simply replace  $\mathcal{A}$  with  $\mathcal{A}^t$  and conjugate it by an appropriate permutation matrix). We then find linearly independent matrices  $U$  and  $V$  in  $M_{p,q}(\mathbb{K})$  such that  $\mathcal{A}$  is generated by

$$\begin{bmatrix} I_p & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & I_q \end{bmatrix}, \begin{bmatrix} 0 & U \\ 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & V \\ 0 & 0 \end{bmatrix}.$$

The considerations of Section 2 show that the kernel of

$$g : \begin{cases} M_p(\mathbb{K}) \times M_q(\mathbb{K}) & \longrightarrow M_{p,q}(\mathbb{K})^2 \\ (X, Y) & \longmapsto (XU - UY, XV - VY) \end{cases}$$

has a dimension greater than 1 if and only if  $\mathcal{C}(\mathcal{A})$  is non-trivial.

**Lemma 9.** *With the preceding notations, for to have  $\dim \text{Ker } g = 1$ , it is necessary that:*

- either  $q = p$  and at least one of the matrices  $U$  or  $V$  has rank  $p$ ;
- or  $q = p + 1$  and both matrices  $U$  and  $V$  have rank  $p$ .

In order to prove this lemma, we will start with another one:

**Lemma 10.** *Let  $W \in M_{p,q}(\mathbb{K})$  be such that  $\text{rk } W < p \leq q$ . Then the linear map*

$$f : \begin{cases} M_p(\mathbb{K}) \times M_q(\mathbb{K}) & \longrightarrow M_{p,q}(\mathbb{K}) \\ (X, Y) & \longmapsto XW - WY \end{cases}$$

*is not onto.*

*Proof.* Notice that  $\text{rk } f$  is unchanged by replacing  $W$  with an equivalent matrix: for every  $(P_1, Q_1) \in \text{GL}_p(\mathbb{K}) \times \text{GL}_q(\mathbb{K})$  and  $(X, Y) \in M_p(\mathbb{K}) \times M_q(\mathbb{K})$ , one can indeed write:

$$XP_1WQ_1 - P_1WQ_1Y = P_1 \left[ (P_1^{-1}XP_1)W - W(Q_1YQ_1^{-1}) \right] Q_1.$$

Letting  $r := \text{rk } W$ , we then lose no generality assuming that  $W = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ . In this case however, every matrix of  $\text{Im } f$  has a zero entry in position  $(p, q)$ , hence  $f$  is not onto.  $\square$

*Proof of Lemma 9.* setting  $f_1 : (X, Y) \mapsto XU - UY$  and  $f_2 : (X, Y) \mapsto XV - VY$ , we find that  $\text{rk } g \leq \text{rk } f_1 + \text{rk } f_2$  whilst Lemma 10 shows that  $\text{rk } f_1 + \text{rk } f_2 \leq 2M_{p,q}(\mathbb{K}) - m$  where  $m$  is the number of matrices in  $\{U, V\}$  which have rank lesser than  $p$ . We deduce that  $\text{rk } g \leq 2pq - m$ . The rank theorem then shows that

$$\dim \text{Ker } g \geq p^2 + q^2 - 2pq + m = (q - p)^2 + m.$$

If  $q \geq p + 2$ , then  $\dim \text{Ker } g \geq 4$ . If  $q = p + 1$  and  $m \geq 1$ , then  $\dim \text{Ker } g \geq 2$ . If  $q = p$  and  $m = 2$ , then  $\dim \text{Ker } g \geq 2$ . This proves the claimed results.  $\square$

We may now complete the proof of Theorem 2. Assume  $n$  is even. Then Lemma 9 shows  $\mathcal{C}(\mathcal{A})$  is non-trivial unless  $p = q = \frac{n}{2}$  and one of the matrices  $U$  and  $V$  is non-singular.

Assume then  $p = q = \frac{n}{2}$  and  $U$  is non-singular (for example). Conjugating  $\mathcal{A}$

by  $\begin{bmatrix} I_p & 0 \\ 0 & U \end{bmatrix}$ , we are then reduced to the case  $U = I_p$ : in this case  $V \notin \text{Span}(I_p)$  and the matrix  $\begin{bmatrix} V & 0 \\ 0 & V \end{bmatrix}$  clearly belongs to  $\mathcal{C}(\mathcal{A}) \setminus \text{Span}(I_n)$ .

Let us finish this section by summing up the results in the case  $\dim \mathcal{A} = 4$ . Recall that we have assumed that  $\mathbb{K}$  is algebraically closed or that  $\mathcal{A}$  contains a non-trivial idempotent.

- (i) If  $n$  is even, then  $\mathcal{C}(\mathcal{A})$  is non-trivial.
- (ii) If  $n$  is odd and  $\mathcal{C}(\mathcal{A})$  is trivial, then, setting  $p := \frac{n-1}{2}$ , there are linearly independent matrices  $U$  and  $V$  in  $M_{p,p+1}(\mathbb{K})$  with rank  $p$  and a non-singular matrix  $P \in \text{GL}_n(\mathbb{K})$  such that either

$$P \mathcal{A} P^{-1} = \left\{ \begin{bmatrix} a.I_p & c.U + d.V \\ 0 & b.I_{p+1} \end{bmatrix} \mid (a, b, c, d) \in \mathbb{K}^4 \right\}$$

or

$$P \mathcal{A}^t P^{-1} = \left\{ \begin{bmatrix} a.I_p & c.U + d.V \\ 0 & b.I_{p+1} \end{bmatrix} \mid (a, b, c, d) \in \mathbb{K}^4 \right\}.$$

This of course completes the proof of Theorem 2.

## 4 On subalgebras of dimension 4 of $M_{2n+1}(\mathbb{K})$ with a trivial centralizer

Here, we establish Proposition 5 by prolonging the proof of Section 3.6 in the case  $\dim \mathcal{A} = 4$ . We must first return to the situation where  $\mathbb{K}$  is an arbitrary field.

**Lemma 11.** *Let  $n \in \mathbb{N}^*$  and  $\mathcal{A}$  be a 4-dimensional subalgebra of  $M_{2n+1}(\mathbb{K})$  with a trivial centralizer. Then  $\mathcal{A}$  contains a rank  $n$  idempotent.*

*Proof.* Choose an algebraically closed extension  $\mathbb{L}$  of  $\mathbb{K}$ . Then  $\mathcal{A}_{\mathbb{L}}$  has a trivial centralizer in  $M_{2n+1}(\mathbb{L})$ . The proof in Sections 3.2, 3.3, 3.4 and 3.6 then shows that there is a 2-dimensional subspace  $P \subset M_{n,n+1}(\mathbb{K})$  such that  $\mathcal{A}_{\mathbb{L}}$  is conjugate to either the subalgebra

$$\mathcal{H} := \left\{ \begin{bmatrix} a.I_n & M \\ 0 & b.I_{n+1} \end{bmatrix} \mid (a, b) \in \mathbb{L}^2, M \in P \right\}$$

or its transpose  $\mathcal{H}^t$ . In any case, the set of unispectral elements in  $\mathcal{A}_{\mathbb{L}}$  is a linear hyperplane of  $\mathcal{A}_{\mathbb{L}}$ : this is the case indeed when  $\mathcal{A}_{\mathbb{L}} = \mathcal{H}$  since this subset is then

$$\left\{ \begin{bmatrix} a.I_n & M \\ 0 & a.I_{n+1} \end{bmatrix} \mid a \in \mathbb{L}, M \in P \right\}.$$

Also, every non-unispectral element of  $\mathcal{A}_{\mathbb{L}}$  is clearly diagonalisable with exactly two eigenvalues of respective orders  $n$  and  $n+1$ .

Every basis of the  $\mathbb{K}$ -vector space  $\mathcal{A}$  is also a basis of the  $\mathbb{L}$ -vector space  $\mathcal{A}_{\mathbb{L}}$  and therefore must contain a matrix which is not unispectral in  $M_n(\mathbb{L})$ . Let us choose such a matrix  $M \in \mathcal{A}$ , with eigenvalues  $\lambda$  and  $\mu$  of respective orders  $n$  and  $n+1$ . Notice then that  $\lambda$  and  $\mu$  belong to  $\mathbb{K}$ . Indeed:

- the minimal polynomial of  $M$  is  $X^2 - (\lambda + \mu)X + \lambda\mu$ , so  $\lambda + \mu \in \mathbb{K}$  (implicit here is the fact that the minimal polynomial of a matrix is unchanged by extending the field of scalars);
- also  $\text{tr}(M) = n(\lambda + \mu) + \mu$ , which entails  $\mu \in \mathbb{K}$  and therefore  $\lambda \in \mathbb{K}$ .

We deduce that  $\frac{1}{\lambda - \mu} (M - \mu.I_{2n+1})$  is a rank  $n$  idempotent in  $\mathcal{A}$ .  $\square$

Proposition 5 can now be proven. Since  $\mathcal{A}$  contains an idempotent of rank  $n$ , the proof from Sections 3.4 and 3.6 shows that we can reduce the study to the situation where there is a 2-dimensional subspace  $\mathcal{P} \subset M_{n,n+1}(\mathbb{K})$  such that

$$\mathcal{A} = \left\{ \begin{bmatrix} a.I_n & M \\ 0 & b.I_{n+1} \end{bmatrix} \mid (a, b) \in \mathbb{K}^2, M \in \mathcal{P} \right\}.$$

Let  $U \in \mathcal{P} \setminus \{0\}$  and choose  $V$  such that  $(U, V)$  is a basis of  $\mathcal{P}$ . Then Lemma 11 shows that  $U$  (and  $V$ ) must have rank  $n$ . For every extension  $\mathbb{L}$  of  $\mathbb{K}$ , the subalgebra  $\mathcal{A}_{\mathbb{L}}$  has a trivial centralizer, which shows  $\text{rk } U = n$  for every  $U \in \mathcal{P}_{\mathbb{L}} \setminus \{0\}$ . We will then use the following result to see that  $\mathcal{P}$  is equivalent to the 2-dimensional subspace  $\text{Span}(C_n, D_n)$  i.e.  $\mathcal{P} = P \text{Span}(C_n, D_n) Q$  for some pair  $(P, Q) \in \text{GL}_n(\mathbb{K}) \times \text{GL}_{n+1}(\mathbb{K})$ :

**Proposition 12.** *Let  $A$  and  $B$  in  $M_{n,n+1}(\mathbb{K})$ . Assume that every non-trivial linear combination of  $A$  and  $B$  over any field extension of  $\mathbb{K}$  has rank  $n$ . Then there is a pair  $(P, Q) \in \text{GL}_n(\mathbb{K}) \times \text{GL}_{n+1}(\mathbb{K})$  such that  $A = P C_n Q^{-1}$  and  $B = P D_n Q^{-1}$ .*

Assuming this to be true, consider a basis  $(A, B)$  of  $\mathcal{P}$  and a pair  $(P, Q)$  associated to it as in Proposition 12. Then a straightforward computation shows that  $R \mathcal{H}_{2n+1}(\mathbb{K}) R^{-1} = \mathcal{A}$  for  $R = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix}$ , which completes the proof of Theorem 5.

*Proof of Proposition 12.* We let  $x$  denote an indeterminate. We use the Kronecker-Weierstrass reduction for pencils of matrices (see chapter XII of [2] and the appendix of [4] for the case of an arbitrary field). Since  $A$  and  $B$  have rank  $n$  and belong to  $M_{n,n+1}(\mathbb{K})$ , the canonical form of the pencil  $A + xB$  cannot contain any block of the forms

$$\begin{bmatrix} x & 0 & & & \\ 1 & x & 0 & & \\ 0 & 1 & x & & \\ & & \ddots & \ddots & \\ & & & \ddots & x \\ & & & & 1 \end{bmatrix} \in M_{p+1,p}(\mathbb{K}[x]) \quad ; \quad \begin{bmatrix} 1 & x & 0 & & \\ 0 & 1 & x & & \\ & & \ddots & \ddots & \\ & & & \ddots & x \\ & & & & 1 \end{bmatrix} \in M_p(\mathbb{K}[x])$$

nor

$$\begin{bmatrix} x & 1 & 0 & & \\ 0 & x & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & x \end{bmatrix} \in M_p(\mathbb{K}[x])$$

and contains at most one block of the form

$$L_p := \begin{bmatrix} 1 & x & 0 & & \\ 0 & 1 & x & & \\ & & \ddots & \ddots & \\ & & & 1 & x \end{bmatrix} \in M_{p,p+1}(\mathbb{K}[x]).$$

It follows that there exists some  $p \in \llbracket 0, n-1 \rrbracket$ , some non-singular  $C \in GL_p(\mathbb{K})$  and some pair  $(P, Q) \in GL_n(\mathbb{K}) \times GL_{n+1}(\mathbb{K})$  such that

$$P^{-1}(A + xB)Q = \begin{bmatrix} C + x.I_p & 0 \\ 0 & L_{n-p} \end{bmatrix}.$$

However, if  $p > 0$ , then  $\text{rk}(A - \lambda B) < n$  for any eigenvalue  $\lambda$  of  $C$ , which contradicts the assumptions. It follows that  $p = 0$ , hence  $P^{-1}(A + xB)Q = L_n = C_n + xD_n$ , which shows  $A = PC_nQ^{-1}$  and  $B = PD_nQ^{-1}$ .  $\square$

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## References

- [1] L. Chambadal, J.L. Ovaert. *Algèbre linéaire et algèbre tensorielle*. Dunod, 1968.
- [2] F.A. Gantmacher. *The Theory of Matrices, vol. 2*. Chelsea Publishing Company, 1959.
- [3] N. Jacobson, *Basic Algebra II*, 3rd edition. W.H. Freeman & Company, 1980.
- [4] C. de Seguins Pazzis, Invariance of simultaneous similarity and equivalence of matrices under extension of the ground field, *Linear Algebra Appl.*, **433-3** (2010), 618-624.
- [5] D.A. Suprunenko, R.I. Tyshkevich. *Commutative Matrices*. Academic Press, 1968.